ESTIMATION OF PARAMETERS FROM GENERALISED CENSORED NORMAL SAMPLES

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1. Introduction

The problem of estimating the parameters from censored samples has been considered by Hald (1949), Cohen (1950), Gupta (1952), Desraj (1953) and others for a number of distributions. In actual practice the censoring of the samples may be different from that considered by these authors. For example, suppose we are interested in the study of the life of, say, clothing items. Let these items be issued to a military unit, on a particular date, out of which k failures are reported to the condemnation board after a fixed period. If now during this period a certain number of soldiers, say, $n_1, n_2, \ldots, n_l$ is transferred to other units at different times $t_1', t'_2, \ldots, t'_l$ for some strategic reasons or otherwise, it means that information regarding the life of the items used by these soldiers is not easily available from the records. The only information available is that these items have been in use up to the time of their transfer. We may treat such type of data as a sample from a normal population censored at various stages.

The problem of estimating the parameters of a normal population from a sample censored at various stages has been discussed in this paper by the method of maximum likelihood. The expectations of the second-order partial derivatives of maximum likelihood expressions have been worked out to determine the standard errors (s.e.'s) of the estimates. For the special case of two-stage censored samples, the variance and co-variance of the estimates in terms of $\hat{\sigma}^2/N$ have been obtained and given in the Appendix where $\hat{\sigma}^2$ is the estimate of σ^2 and N is the sample size. A similar problem for exponential populations involving only a single parameter has been discussed in detail by Bartholomew (1957).

2. Maximum Likelihood Solutions for Generalised Censored Samples

(2.*a*) *Explanation of Notations and Symbols*

(i) Number of items issued: N

(ii) Period of observation: t_k

(iii) Number of items condemned (or failed) by the timc interval t_k: k

(iv) Time elasped before j-th item fails or gets condemned: t_j, $j = 1, 2, \ldots, k$

(v) Number of subjects leaving the unit or regiment during interval t_k before failure: n

(vi) Time points at which n_j subjects leave the unit: t'_j

(vii) Number of items surviving up to time t_k: $N - k - n$

(viii) k_j is the number of failures observed during the time interval t'_j.

(ix) $n = \sum_{j=1}^{l} n_j$

(x) $\nu_\omega = \frac{1}{k} \sum_{j=1}^{k} (t_j - t_k)^\omega, \quad \omega = 1, 2$

(xi) $\theta = \frac{1}{k} \sum_{j=1}^{l} n_j A_j$

(xii) $p_T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta_T} e^{-\frac{1}{2}\eta^2} d\eta$

(xiii) $p_j = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta_j} e^{-\frac{1}{2}\eta^2} d\eta$

(xiv) $\phi = \frac{1}{k} \sum_{j=1}^{l} n_j A_j \eta'_j$

where

$$A_j = \frac{f(\eta'_j)}{F(\eta'_j)}$$

$f(\eta)$ is the probability density for normal distribution, given by

$$\frac{1}{\sqrt{2\pi}} e^{-\eta^2/2}, \quad -\infty \leqslant \eta \leq \infty$$

$$F(\eta_j) = \int_{\eta_j}^{\infty} f(\eta)\, d\eta, \quad \eta_j = \frac{t_i - m}{\sigma}, \quad \eta_T = \frac{t_k - m}{\sigma}.$$

(2.*b*) *Estimating Equations*

The likelihood function for such a sample is

$$L = \text{const.} \frac{1}{\sigma^k} [F(\eta_T)]^{N-K-n} \prod_{j=1}^{l} [F(\eta'_j)]^{n_j} e^{-\sum_{j=1}^{k} (t_i - m)^2/2\sigma^2}. \tag{2.b.1}$$

First partial derivatives of the logarithm of the expression in the equation (2.*b*.1) with respect to m and σ are

$$\frac{\partial \log L}{\partial m} = \frac{1}{\sigma} \sum_{j=1}^{k} \frac{t_j - m}{\sigma} + \frac{1}{\sigma} \sum_{j=1}^{l} n_j A_j + \frac{N-k-n}{\sigma} A_T, \tag{2.b.2}$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{k}{\sigma} + \frac{1}{\sigma} \sum_{j=1}^{k} \left(\frac{t_j - m}{\sigma}\right)^2 + \frac{1}{\sigma} \sum_{j=1}^{l} n_j A_j \eta'_j + \frac{N-k-n}{\sigma} A_T \eta_T. \tag{2.b.3}$$

On equating to zero, equations (2.*b*.2) and (2.*b*.3) may be re-written as

$$\nu_1 = -\sigma \left[\eta_T + \frac{N-k-n}{k} A_T + \frac{1}{k} \sum_{j=1}^{l} n_j A_j \right] \tag{2.b.4}$$

$$\nu_2 = \sigma^2 \left[1 + \eta_T \left\{ \eta_T + \frac{N-k-n}{k} A_T \right\} + \frac{1}{k} \sum_{j=1}^{l} n_j A_j (2\eta_T - \eta'_j) \right] \tag{2.b.5}$$

from equations (2.*b*.4) and (2.*b*.5) we get the estimating equations as

$$\hat{m} = \bar{t} + \frac{\hat{\sigma}}{k} \{k\theta + (N - k - n) A_T\}, \tag{2.b.6}$$

and

$$\frac{\nu_2}{\hat{\sigma}^2} = 1 - \phi + \eta_T \left(\theta - \frac{\nu_1}{\hat{\sigma}}\right) \tag{2.b.7}$$

where $\bar{t}$ is the sample mean and $\hat{m}$ and $\hat{\sigma}$ are the estimates. By solving the equation (2.*b*.7), σ can be estimated, where first estimates of η's are obtained from p_T and p_j's by using normal probability tables and taking $p_T = k/(N - n)$ and $p_j = k_j/(N - \sum_{r=1}^{j} n_r)$. Knowing σ, m can be estimated from (2.*b*.6). By repeating this process a number of times the best estimates of m and σ may be evaluated. When $n_1, n_2, \ldots, n_l$ are all zero the equations (2.*b*.6) and (2.*b*.7) reduce to the ordinary (one point) censored case discussed by Gupta and Cohen.

3. Precision of the Estimates

The large sample variances and co variances of the estimates may be evaluated by calculating the expected values of

$$-\frac{\partial^2 \log L}{\partial m^2}$$

$$= \frac{k}{\sigma^2} + \frac{1}{\sigma^2} \sum_{j=1}^{l} n_j A_j (A_j - \eta'_j) + \left(N - k - \sum_{j=1}^{l} n_j\right) \times \frac{A_T (A_T - \eta_T)}{\sigma^2}, \tag{3.1}$$

$$-\frac{\partial^2 \log L}{\partial \sigma^2}$$

$$= -\frac{k}{\sigma^2} + \frac{3}{\sigma^2} \sum_{j=1}^{k} \left(\frac{t_j - m}{\sigma}\right)^2 + \frac{1}{\sigma^2} \sum_{j=1}^{l} n_j \eta'_j A_j \{2 + \eta'_j (A_j - \eta'_j)\}$$
$$+ \left(N - k - \sum_{j=1}^{l} n_j\right) \frac{1}{\sigma^2} \eta_T A_T \{2 + \eta_T (A_T - \eta_T)\}, \tag{3.2}$$

$$-\frac{\partial^2 \log L}{\partial m \partial \sigma}$$

$$= \frac{2}{\sigma^2} \sum_{j=1}^{k} \frac{t_j - m}{\sigma} + \frac{1}{\sigma^2} \sum_{j=1}^{l} n_j A_j \{1 + \eta'_j (A_j - \eta'_j)\}$$

$$+ \left(N - k - \sum_{j=1}^{l} n_j \right) \{1 + \eta_T (A_T - \eta_T)\}. \qquad (3.3)$$

For large sample size, p_T being fixed, n_j being known, the expected values (3.1), (3.2) and (3.3) are given below:

$$E\left(- \frac{\partial^2 \log L}{\partial m^2} \right)$$

$$= \frac{N}{\sigma^2} \left[p_T + f(\hat{\eta}_T)(A_T - \hat{\eta}_T) + \sum_{j=1}^{l} \frac{n_j}{N} \{-(p_T - p_j) + \theta_j + a_j \theta_T\} \right]$$

$$= \frac{N}{\sigma^2} I_{11} \qquad (3.4)$$

$$E\left(- \frac{\partial^2 \log L}{\partial \sigma^2} \right)$$

$$= \frac{N}{\sigma^2} \left[2p_T - \hat{\eta}_T f(\hat{\eta}_T) + \hat{\eta}_T^2 f(\hat{\eta}_T)(A_T - \hat{\eta}_T) \right.$$

$$+ \sum_{j=1}^{l} \frac{n_j}{N} \left\{ -2(p_T - p_j) + \frac{3}{p_T} (\hat{\eta}_T f(\hat{\eta}_T)(p_T - p_j) \right.$$

$$\left. \left. + \hat{\eta}_T p_j f(\hat{\eta}_T) - p_T \hat{\eta}'_j f(\hat{\eta}'_j)) + \phi_j + a_j \phi_T \right\} \right]$$

$$= \frac{N}{\sigma^2} I_{22} \qquad (3.5)$$

$$E\left(- \frac{\partial^2 \log L}{\partial \sigma \partial m} \right)$$

$$= \frac{N}{\sigma^2} \left[-f(\hat{\eta}_T) + \hat{\eta}_T f(\hat{\eta}_T)(A_T - \hat{\eta}_T) \right.$$

$$+ \sum_{j=1}^{l} \frac{n_j}{N} \left\{ 2 (p_T - p_j) \frac{f(\hat{\eta}_T)}{p_T} + \psi_j + \alpha_j \psi_T \right\} \Bigg]$$

$$= \frac{N}{\sigma^2} I_{12} \qquad (3.6)$$

where

$$\left. \begin{array}{l} \alpha_j = p_T - p_j - 1, \quad \theta_j = A_j (A_j - \hat{\eta}'_j) \\ \phi_j = \hat{\eta}'_j A_j \{2 + \hat{\eta}'_j (A_j - \hat{\eta}'_j)\} \\ \psi_j = A_j \{1 + \hat{\eta}'_j (A_j - \hat{\eta}'_j)\} \end{array} \right\} . \qquad (3.7)$$

They have been obtained by using the following relations:

$$\left. \begin{array}{l} E(k) = Np_T - \sum_{j=1}^{l} n_j (p_T - p_j) \\ E(N-k) = N(1-p_T) + \sum_{j=1}^{l} n_j (p_T - p_j) \\ E\left(N - k - \sum_{j=1}^{l} n_j\right) = N(1-p_T) + \sum_{j=1}^{l} n_j (p_T - p_j) - \sum_{j=1}^{l} n_j \\ E\left(\frac{t-m}{\sigma}\right) = \frac{1}{p} \int_{-\infty}^{\hat{\eta}} \eta f(\eta)\, d\eta = -\frac{1}{p} f(\hat{\eta}) \\ E\left(\frac{t-m}{\sigma}\right)^2 = \frac{1}{p} \int_{-\infty}^{\hat{\eta}} \eta^2 f(\eta)\, d\eta = 1 - \frac{1}{p} \hat{\eta} f(\hat{\eta}) \end{array} \right\} \qquad (3.8)$$

where $\hat{\eta}$ is the solution of $\int_{-\infty}^{\hat{\eta}} f(\eta)\, d\eta = p$.

The variances and covariances of the estimates in terms of $\hat{\sigma}^2/N$ can be determined by inverting the usual information matrix.

$$(I_{ij}) = \begin{bmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{bmatrix} . \qquad (3.9)$$

Hence the variance-covariance matrix of the estimates is given in terms of $\hat{\sigma}^2/N$ as $(\sigma_{ij}) = (I_{ij})^{-1}$. A few values of σ_{ij} $(i, j = 1, 2)$ for $p_T = 0{\cdot}3\ (0{\cdot}1)\ 0{\cdot}9$, $p_j = 0{\cdot}1\ (0{\cdot}1)\ (p_T - 0{\cdot}1)$ and $n_j/N = 0{\cdot}1$

* n_j/N cannot exceed $(1 - p_T)$.

$(0 \cdot 1)\ (1 - p_T)^*$ are given in the tables (appended) for two-stage censored samples. These tables can be extended for more values of p_j and several stages of censoring.

4. Numerical Example

A random sample of size three hundred was taken from the tables of random samples for a normal population by Mahalanobis (1934) with $m = 10$ and $\sigma = 1$. Let the number of observations n_1 and n_2 not available at $t'_1 = 8 \cdot 438$, $t'_2 = 9 \cdot 496$ be 5 and 10 respectively. Taking $t_k = 9 \cdot 709$ and $k = 100$, the sample yields the following:—

$$t = 8 \cdot 94142$$

$$\nu_1 = - \cdot 76758,\ \nu_2 = 0 \cdot 94025,\ n = n_1 + n_2 = 15.$$

Starting with the observed η's and A's based on (xii) and (xiii) two successive sets of approximations were calculated by iteration using the estimating equations (2.*b*.7) and (2.*b*.6), with the aid of normal probability tables. The averages of these two values can be taken as final estimates of m and σ beause the values of estimating equations (2.*b*.4) and (2.*b*.5) on the basis of average estimates are very small as shown in columns 5 and 6 of the table given below. To check this a number of successive iterations were carried out and the results obtained even in the seventh iteration were very unsatisfactory as compared with the averages of the first two iterations. Standard errors of the estimates are also given in the table.

TABLE

	Iterative estimation of parameters				Values of the estimating equations	
	$\hat{m}$	s.e. of $\hat{m}$	$\hat{\sigma}$	s.e. of $\hat{\sigma}$	(2.*b*.4)	(2.*b*.5)
	1	2	3	4	5	6
1	10.18454	..	1.11742	..	0.05260	0.01201
2	10.15205	..	1.13629	..	0.04325	0.01293
Average of 1 and 2	10.16830	0.10991	1.12686	0.10357	0.00479	0.00035

SUMMARY

The paper gives maximum likelihood equations for estimating the parameters of a normal population from a sample censored at various stages. The variance-covariance matrix has been obtained for calculating the s.e.'s of the estimates. A table giving $\sigma_{ij} (i, j = 1, 2)$ in terms of $\hat{\sigma}^2/N$ has been prepared for two-stage censored samples. A numerical example is discussed to show the practical application.

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APPENDIX

Variances and covariances of maximum likelihood estimates in terms of $\hat{\sigma}^2/N$ for two-stage censored samples

p_1	$\frac{n_1}{N}$	p_T = 0·3			p_T = 0·4		
		σ_{11}	σ_{12}	σ_{22}	σ_{11}	σ_{12}	σ_{22}
	0·05	3·10871	1·86681	2·25698	2·06679	1·06112	1·63413
	0·10	3·20338	1·90388	2·26710	2·14881	1·09914	1·65413
	0·20	3·41379	1·98688	2·29117	2·33427	1·18505	1·69872
0·1	0·30	3·65737	2·08369	2·32114	2·55482	1·28710	1·75077
	0·40	3·94247	2·19770	2·35831	2·82171	1·41054	1·81276
	0·50	4.28068	2·33371	2·40473	3·15102	1·56269	1·88797
	0·60	4·68820	2·49849	2·46321	3·56771	1·75509	1·98176
	0·70	5·18857	2·70167	2·53766	..	..	..
	0·05	3·06992	1·85740	2·25950	2·04613	1·05865	1·63742
	0·10	3·12173	1·88363	2·27168	2·10457	1·09325	1·66050
	0·20	3·23080	1·93884	2·29741	2·23232	1·16898	1·71064
0·2	0·30	3·34811	1·99831	2·32536	2·37669	1·25467	1·76679
	0·40	3·47421	2·06223	2·35556	2·54125	1·35244	1·83023
	0·50	3·61073	2·13154	2·38856	2·73054	1·46501	1·90258
	0·60	3·75867	2·20669	2·42456	2·95060	1·59601	1·98601
	0·70	3·91936	2·28831	2·46384	..	..	..
	0·05	..	..	..	2·02218	1·04857	1·63451
	0·10	..	..	..	2·05457	1·07199	1·65417
	0·20	..	..	..	2·12287	1·12151	1·69552
0·3	0·30	..	..	..	2·19629	1·17490	1·73985
	0·40	..	..	..	2·27533	1·23251	1·78742
	0·50	..	..	..	2·36074	1·29492	1·83865
	0·60	..	..	..	2·45326	1·36271	1·89400
	0·70	..	..	..	..	..	..

p_1	$\frac{n_1}{N}$	p_T = 0·5 σ_{11}	σ_{12}	σ_{22}	p_T = 0·6 σ_{11}	σ_{12}	σ_{22}
	0·05	1·58067	0·63658	1·26403	1·32680	0·38667	1·01888
	0·10	1·64989	0·67081	1·28818	1·38602	0·41637	1·04483
0·1	0·20	1·80855	0·74955	1·34203	1·52314	0·48616	1·10328
	0·30	2·00146	0·84577	1·40521	1·69266	0·57406	1·17296
	0·40	2·24109	0·96584	1·48102	1·90777	0·68762	1·25827
	0·50	2·54680	1·11966	1·57454	..	..	..
	0·05	1·56753	0·63764	1·26881	1·31703	0·38954	1·02504
	0·10	1·62174	0·67265	1·29803	1·36528	0·42231	1·05789
0·2	0·20	1·74345	0·75180	1·36299	1·47593	0·49894	1·13288
	0·30	1·88683	0·84588	1·43858	1·61091	0·59467	1·22375
	0·40	2·05832	0·95938	1·52790	1·77968	0·71714	1·33669
	0·50	2·26725	1·09874	1·63547	..	..	..
	0·05	1·55384	0·63420	1·27071	1·30790	0·38929	1·02990
	0·10	1·59284	0·66513	1·30175	1·34603	0·42169	1·06813
0·3	0·20	1·67854	0·73375	1·36994	1·43281	0·49711	1·15583
	0·30	1·77639	0·81301	1·44774	1·53756	0·59071	1·26269
	0·40	1·88931	0·90551	1·53747	1·66709	0·70956	1·39603
	0·50	2·02119	1·01471	1·64219	..	..	..
	0·05	1·53763	0·62422	1·26328	1·29815	0·38522	1·02894
	0·10	1·55896	0·64404	1·28601	1·32545	0·41283	1·06588
0·4	0·20	1·60425	0·68643	1·33441	1·38648	0·47584	1·14948
	0·30	1·65345	0·73289	1·38717	1·45820	0·55182	1·24925
	0·40	1·70710	0·78403	1·44493	1·54408	0·64504	1·37046
	0·50	1·76590	0·84055	1·50846	..	..	..
	0·05	..	..	..	1·28677	0·37601	1·01859
	0·10	..	..	..	1·30150	0·39310	1·04370
0·5	0·20	..	..	..	1·33318	0·43033	1·09817
	0·30	..	..	..	1·36825	0·47223	1·15920
	0·40	..	..	..	1·40733	0·51968	1·22803
	0·50	..	..	..	..	..	..

p_1	$\frac{n_1}{N}$	p_T 0·7 σ_{11}	0·7 σ_{12}	0·7 σ_{22}	0·8 σ_{11}	0·8 σ_{12}	0·8 σ_{22}
	0·05	1·18557	0·22937	0·84456	1·10477	0·12641	0·71419
0·1	0·10	1·23747	0·25494	0·87146	1·15141	0·14866	0·74214
	0·20	1·35845	0·31649	0·93293	1·26076	0·20377	0·80728
	0·30	1·50991	0·39669	1·00784	..	..	..
	0·05	1·11739	0·23324	0·85208	1·09734	0·13100	0·72343
0·2	0·10	1·22029	0·26328	0·88774	1·13604	0·15885	0·76258
	0·20	1·32048	0·33616	0·97164	1·22811	0·22960	0·85839
	0·30	1·44673	0·43246	1·07829	..	..	..
	0·05	1·17048	0·23470	0·85913	1·09155	0·13360	0·73273
0·3	0·10	1·20586	0·26658	0·90316	1·12422	0·16499	0·78363
	0·20	1·28909	0·34479	1·00906	1·20391	0·24737	0·91396
	0·30	1·39542	0·45017	1·14835	..	..	..
	0·05	1·16379	0·23372	0·86242	1·08643	0·13453	0·73956
0·4	0·10	1·19182	0·26445	0·91031	1·11366	0·16739	0·79939
	0·20	1·25776	0·33981	1·02628	1·18186	0·25578	0·95775
	0·30	1·34226	0·44143	1·18036	..	..	..
	0·05	1·15666	0·22959	0·85927	1·08141	0·13341	0·74186
0·5	0·10	1·17661	0·25524	0·90312	1·10309	0·16491	0·80465
	0·20	1·22276	0·31653	1·00713	1·15805	0·24977	0·97208
	0·30	1·28010	0·39584	1·14059	..	..	..
	0·05	1·14842	0·22113	0·84652	1·07607	0·12945	0·73707
0·6	0·10	1·15910	0·23676	0·87521	1·09154	0·15571	0·79324
	0·20	1·18258	0·27176	0·93926	1·12996	0·22375	0·93806
	0·30	1·20948	0·31281	1·01410	..	..	..
	0·05	..	..	..	1·06994	0·12138	0·72148
0·7	0·10	..	..	..	1·07818	0·13739	0·75773
	0·20	..	..	..	1·09721	0·17514	0·84304
	0·30	..	..	..	..	..	..

p_1	$\frac{n_1}{N}$	p_T 0·9		
		σ_{11}	σ_{12}	σ_{22}
0·1	0·05	1·05946	0·05854	0·61346
	0·10	1·10253	0·07870	0·64407
0·2	0·05	1·05237	0·06410	0·62591
	0·10	1·08816	0·09155	0·67249
0·3	0·05	1·04721	0·06801	0·63892
	0·10	1·07800	0·10132	0·70353
0·4	0·05	1·04293	0·07074	0·65054
	0·10	1·06968	0·10865	0·73242
0·5	0·05	1·03905	0·07205	0·65894
	0·10	1·06198	0·11256	0·75421
0·6	0·05	1·03524	0·07139	0·66200
	0·10	1·05396	0·11121	0·76214
0·7	0·05	1·03114	0·06772	0·65627
	0·10	1·04473	0·10179	0·74676
0·8	0·05	1·02636	0·05916	0·63574
	0·10	1·03350	0·08046	0·69509

p_1 and p_T are to be noted from XII and XIII of Section (2.*a*) on the basis of final estimates of η_1 and η_T.

p_1 and p_t are the proportions corresponding to the first and second points of truncation respectively and n_1 the number of units leaving at the first point.